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A Multiplication of Distributions

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If Ω denotes an open subset of $\mathbb{R}^n$ $(n = 1, 2,...)$, we define an algebra $\mathscr{G}(\Omega)$ which contains the space $\mathscr{D}'(\Omega)$ of all distributions on Ω and such that $C^\infty(\Omega)$ is a subalgebra of $\mathscr{G}(\Omega)$. The elements of $\mathscr{G}(\Omega)$ may be considered as "generalized functions" on Ω and they admit partial derivatives at any order that generalize exactly the derivation of distributions. The multiplication in $\mathscr{G}(\Omega)$ gives therefore a natural meaning to any product of distributions, and we explain how these results agree with remarks of Schwartz on difficulties concerning a multiplication of distributions. More generally if $q = 1, 2,...,$ and $f \in \mathscr{O}_M(\mathbb{R}^{2q})$—a classical Schwartz notation—for any $G_1,..., G_q \in \mathscr{G}(\Omega)$, we define naturally an element $f(G_1,..., G_q) \in \mathscr{G}(\Omega)$. These results are applied to some differential equations and extended to the vector valued case, which allows the multiplication of vector valued distributions of physics.

INTRODUCTION

From the mathematical viewpoint quantum field theory is based on heuristic computations done on "objects" that are treated as if they were usual functions (of domain $\mathbb{R}^n$ and operator valued) from the viewpoint of multiplication, derivation, and integration. It is now well known that the simplest of these "objects" (for instance, the free fields operators) are distributions, but most of these "objects," written for instance as "products" of distributions, remain completely mysterious.

Remarks of Schwartz [13] prove that, if one has an associative multiplication with the usual formula for the derivative of a product, one is obliged to abandon either the usual concept of derivation of C^1 functions or the usual concept of multiplication of continuous functions (let us note there is no problem concerning C^∞ functions). The derivation in the sense of distributions is clearly used in quantum field theory so we shall keep it, and as a consequence our multiplication will only generalize the multiplication of

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C^∞ functions, that is to say $C^\infty(\Omega)$ will be a subalgebra of $\mathcal{G}(\Omega)$ but the algebra $\mathscr{C}(\Omega)$ of all continuous functions on Ω will not be a subalgebra of $\mathcal{G}(\Omega)$. This fact might seem troublesome at first: in fact it will not be a difficulty for the following more subtle reason: some elements of $\mathcal{G}(\Omega)$ admit an "associated distribution" and we shall prove that the classical product of several continuous functions is the distribution associated with their new product in $\mathcal{G}(\Omega)$. In this way the classical computations on continuous functions will agree in usual cases with the new computations in $\mathcal{G}(\Omega)$. The same will apply for the usual multiplication of a C^∞ function and a distribution and for more complicated products of distributions used by physicists.

Now we explain the idea that lead us to our concept of generalized functions. One first observes that the elements of the spaces $\mathscr{E}(\mathscr{D}(\Omega))$ and $\mathscr{H}(\mathscr{D}(\Omega))$ of, respectively, the complex valued C^∞ and holomorphic functions on the Schwartz space $\mathscr{D}(\Omega)$ admit partial derivatives at any order in the variable $x \in \Omega$ (defined in Section 1) and may be obviously pointwise multiplied, but from this last viewpoint, are not a generalization of the concept of a function on Ω (since $\int f_1(x)\,\varphi(x)\,dx \int f_2(x)\,\varphi(x)\,dx \neq \int f_1(x) f_2(x)\,\varphi(x)\,dx$, in general). One then observes that the same remarks apply to the spaces $\mathscr{E}(\mathscr{E}'(\Omega))$ and $\mathscr{H}(\mathscr{E}'(\Omega))$ ($\mathscr{E}(\Omega) = C^\infty(\Omega)$ according to Schwartz's notations). At this point let us consider in these last spaces the equivalence relation

$$\Phi_1 \sim \Phi_2 \Leftrightarrow \Phi_1(\delta_x) = \phi_2(\delta_x), \qquad \forall x \in \Omega,$$

where δ_x denotes the Dirac measure at the point x. It is easy to prove that the quotient spaces are algebras isomorphic to the usual algebra $\mathscr{E}(\Omega)$ of all C^∞ functions on Ω. Therefore this shows that, via an adequate quotient, a space which is not acceptable to be considered as a space of—in some sense generalized—functions on Ω may give rise to an excellent space of functions on Ω. Now it looks natural that our algebra $\mathcal{G}(\Omega)$ will be a quotient of a subalgebra of $\mathscr{E}(\mathscr{D}(\Omega))$.

To simplify the notations we shall consider in general the case of a single real variable and even the case $\Omega = \mathbb{R}$; the extension to an arbitrary open subset Ω of $\mathbb{R}^n$ is immediate. We always use the classical notations of Schwartz's distribution theory [12]. For differential calculus and holomorphy in locally convex spaces we refer to [2]. In Section 1 we describe or obtain a few useful properties of the C^∞ functions over $\mathscr{E}'(\mathbb{R})$ and $\mathscr{D}(\mathbb{R})$, in particular we define their partial derivatives in the variable $x \in \mathbb{R}$. In Section 2 we define our space $\mathcal{G}(\mathbb{R})$ of "generalized functions" and we prove the inclusion of $\mathscr{D}'(\mathbb{R})$ into $\mathcal{G}(\mathbb{R})$. In Section 3 we study the relation (already alluded to) between the usual product of the continuous functions and their new product in $\mathcal{G}(\mathbb{R})$. We also study the case of the multiplication of distributions by C^∞ functions. In Section 4 we recover in the same way more complicated products of distributions used in physics and already studied by several mathematicians [1, 3–6, 8–11]. In Section 5 we show that if $q = 1, 2, \ldots, f \in \mathcal{O}_M(\mathbb{R}^{2q})$ (i.e., f is a C^∞ function on $\mathbb{R}^{2q}$, which is slowly increasing as well as all its derivatives) and if $G_1, \ldots, G_q \in \mathcal{G}(\mathbb{R})$, then $f(G_1, \ldots, G_q)$ is naturally defined as an element of $\mathcal{G}(\mathbb{R})$, which is much more general than the multiplication of the generalized functions and shows the richness of the nonlinear properties of our concept of generalized functions. In Section 6 we solve differential equations $X'(t) = ia(t)X(t)$, where a is a real distribution on $\mathbb{R}$ with compact support; this equation is chosen as a (considerably simplified) model of basic unsolved equations of physics and for its solution we already need an imaginary exponential of a real distribution. In Section 7 we define vector valued generalized functions, which give a meaning to the multiplication of vector valued distributions, in particular of the free fields operators.

1. Some Properties of the C^∞ Functions on $\mathcal{E}'(\mathbb{R})$ and $\mathcal{D}(\mathbb{R})$

If $\Phi \in \mathcal{E}(\mathcal{E}'(\mathbb{R}))$, we denote by $\mathcal{A}\Phi$ the function on $\mathbb{R}$ defined by

$$(\mathcal{A}\Phi)(x) = \Phi(\delta_x),$$

where δ_x is the Dirac measure at point x. It is easy to check that the map $x \to \delta_x$ is C^∞ from $\mathbb{R}$ into $\mathcal{E}'(\mathbb{R})$. Therefore $\mathcal{A}\Phi \in \mathcal{E}(\mathbb{R})$. We have the diagram

$$\begin{array}{ccc} \mathcal{E}(\mathcal{E}'(\mathbb{R})) & & \\ \cup & \searrow^{\mathcal{A}} & \\ L(\mathcal{E}'(\mathbb{R})) & \longrightarrow & \mathcal{E}(\mathbb{R}). \end{array}$$

Let $L(\mathcal{E}'(\mathbb{R})) = \mathcal{E}''(\mathbb{R}) = \mathcal{E}(\mathbb{R})$ since $\mathcal{E}(\mathbb{R})$ is reflexive and therefore the restriction of $\mathcal{A}$ to $L(\mathcal{E}'(\mathbb{R}))$ is in fact the identity map on $\mathcal{E}(\mathbb{R})$, therefore $\mathcal{A}(\mathcal{E}(\mathcal{E}'(\mathbb{R})) = \mathcal{E}(\mathbb{R})$. We define an equivalence relation in $\mathcal{E}(\mathcal{E}'(\mathbb{R}))$ by, if $\Phi_1, \Phi_2 \in \mathcal{E}(\mathcal{E}'(\mathbb{R}))$,

$$\Phi_1 \sim \Phi_2 \Leftrightarrow \Phi_1(\delta_x) = \Phi_2(\delta_x), \qquad \forall x \in \mathbb{R}.$$

Clearly the quotient space is an algebra isomorphic to $\mathcal{E}(\mathbb{R})$. We set, if $q = 1, 2, \ldots,$

$$\mathcal{A}q = \left\{ \varphi \in \mathcal{D}(\mathbb{R}) \text{ with } \int \varphi(x)\, dx = 1 \text{ and } \int (x)^i \varphi(x)\, dx = 0 \right.$$
$$\left. \text{if } 1 \leqslant i \leqslant q \right\}.$$

One checks easily that $\mathscr{A}q$ is nonvoid: let $L_i \in \mathscr{D}'(\mathbb{R})$, $0 \leqslant i \leqslant q$, be defined by

$$L_i(\varphi) = \int (x)^i \varphi(x)\, dx,$$

then the family $\{L_i\}_{0 \leqslant i \leqslant q}$ is free in $\mathscr{D}'(\mathbb{R})$, and spans a linear space $\mathscr{L}$ of dimension $q+1$, which therefore admits a topological supplement $\mathscr{M}$. Therefore

$$\mathscr{D}''(\mathbb{R}) = \mathscr{D}(\mathbb{R}) = \mathscr{L}' \oplus \mathscr{M}'$$

and the first element of the dual basis of $\{L_0, ..., L_q\}$ is in $\mathscr{A}q$. There are also more elementary proofs of the nonvoidness of $\mathscr{A}q$, but this one will be used in forthcoming papers. If $\Phi \in \mathscr{A}q$, $\varepsilon > 0$, and $x \in \mathbb{R}$, we set

$$\varphi_{\varepsilon,x}(\lambda) = \frac{1}{\varepsilon} \varphi\left(\frac{\lambda - x}{\varepsilon}\right).$$

1.1. *Remark.* In the case of $\mathbb{R}^n$, $i \in \mathbb{N}^n$ and we define $\mathscr{A}q$ with $1 \leqslant |i| \leqslant q$. Then we set $\varphi_{\varepsilon,x}(\lambda) = (1/(\varepsilon)^n)\, \varphi((\lambda - x)/\varepsilon)$.

1.2. PROPOSITION. *If $\Phi \in \mathscr{E}(\mathscr{E}'(\mathbb{R}))$ and $\Phi \sim 0$, if $\varphi \in \mathscr{A}q$, then for any compact subset K of $\mathbb{R}$ there are $c > 0$ and $\eta > 0$ such that*

$$|\Phi(\varphi_{\varepsilon,x})| \leqslant C(\varepsilon)^{q+1}$$

if $0 < \varepsilon < \eta$ and $x \in K$.

Proof. From the mean value theorem [2, 1.3.2]

$$\Phi(\varphi_{\varepsilon,x}) - \Phi(\delta_x) \in \bar{\Gamma}_{0 \leqslant t \leqslant 1}\{\Phi'(\delta_x + t(\varphi_{\varepsilon,x} - \delta_x)) \times (\varphi_{\varepsilon,x} - \delta_x)\}.$$

Since $\Phi(\delta_x) = 0$ and since the set $\{\delta_x + t(\varphi_{\varepsilon,x} - \delta_x)\}_{x \in K, 0 \leqslant t \leqslant 1, 0 \leqslant \varepsilon \leqslant 1}$ is bounded in $\mathscr{E}'(\mathbb{R})$, therefore relatively compact, and since ϕ' is continuous ($\mathscr{E}'(\mathbb{R})$ is a Silva space, see [2, 0.6.9 and 1.1.6]) it suffices to prove that there is a bounded subset B of $\mathscr{E}'(\mathbb{R})$ such that

$$\varphi_{\varepsilon,x} - \delta_x \in (\varepsilon)^{q+1} B.$$

If $f \in \mathscr{E}(\mathbb{R})$ and $\varphi \in \mathscr{A}q$,

$$\langle \varphi_{\varepsilon,x} - \delta_x, f \rangle = \int [f(\lambda) - f(x)]\, \varphi_{\varepsilon,x}(\lambda)\, d\lambda$$

since $\int \varphi_{\varepsilon,x}(\lambda)\, d\lambda = 1$. The change of variable $\mu = (\lambda - x)/\varepsilon$ gives

$$\langle \varphi_{\varepsilon,x} - \delta_x, f \rangle = \int (f(x + \varepsilon\mu) - f(x))\, \varphi(\mu)\, d\mu.$$

Taylor's formula applied to f gives

$$\langle \varphi_{\varepsilon,x} - \delta_x, f\rangle = \sum_{k=1}^{q} \frac{(\varepsilon)^k}{k!} f^{(k)}(x) \int (\mu)^k \varphi(\mu)\, d\mu$$

$$+ \int r_x(\varepsilon\mu)(\varepsilon\mu)^{q+1} \varphi(\mu)\, d\mu,$$

where, if supp $\varphi \subset [-a, +a]$

$$|r_x(\varepsilon\mu)| \leqslant \operatorname*{Sup}_{|y-x|\leqslant \varepsilon a} \frac{1}{(q+1)!} |f^{(q+1)}(y)|.$$

Since $\varphi \in \mathscr{A}q$,

$$|\langle \varphi_{\varepsilon,x} - \delta_x, f\rangle| \leqslant (\varepsilon)^{q+1} \operatorname*{Sup}_{|y-x|\leqslant \varepsilon a} \frac{1}{(q+1)!} |f^{(q+1)}(y)| \int |\mu|^{q+1} |\varphi(\mu)|\, d\mu.$$

Let K' be a compact subset of $\mathbb{R}$ which contains K in its interior and let $\eta > 0$ be the distance from K to the complement of K'. We set

$$V = \{f \in \mathscr{E}(\mathbb{R}) \text{ such that } \operatorname*{Sup}_{y\in K'} |f^{(q+1)}(y)| \leqslant 1\}.$$

Here V is a 0 neighborhoud in $\mathscr{E}(\mathbb{R})$. If $x \in K$ and $0 < \varepsilon < \eta/a$, we have

$$|\langle \varphi_{\varepsilon,x} - \delta_x, f\rangle| \leqslant C(\varepsilon)^{q+1}$$

with $C > 0$ independent on $f \in V$; therefore if B is the polar set of V,

$$\varphi_{\varepsilon,x} - \delta_x \in C(\varepsilon)^{q+1} B. \quad \blacksquare$$

We know that $\mathscr{D}(\mathbb{R})$ is contained and dense in $\mathscr{E}'(\mathbb{R})$. Since $\mathscr{E}'(\mathbb{R})$ is a Silva space the C^∞ functions on $\mathscr{E}'(\mathbb{R})$ are continuous ([2, 0.6.9 and 1.1.6]), therefore the restriction map

$$\begin{array}{ccc} \mathscr{E}(\mathscr{E}'(\mathbb{R})) & \xrightarrow{r} & \mathscr{E}(\mathscr{D}(\mathbb{R})) \\ \Phi & & \Phi/\mathscr{D}(\mathbb{R}) \end{array}$$

is injective and we therefore consider that $\mathscr{E}(\mathscr{E}'(\mathbb{R}))$ is contained in $\mathscr{E}(\mathscr{D}(\mathbb{R}))$ via the map r. Now we define the concept of derivative in the variable $x \in \mathbb{R}$ of the elements of $\mathscr{E}(\mathscr{E}'(\mathbb{R}))$ and $\mathscr{E}(\mathscr{D}(\mathbb{R}))$.

1.3. Definition. If $\Phi \in \mathscr{E}(\mathscr{D}(\mathbb{R}))$, we define an element $(d/dx)\Phi$ of $\mathscr{E}(\mathscr{D}(\mathbb{R}))$ by

$$\left(\frac{d}{dx}\Phi\right)(\varphi) = -\Phi'(\varphi)\left(\frac{d}{dx}\varphi\right),$$

where φ ranges over $\mathscr{D}(\mathbb{R})$ and, where $\Phi'(\varphi) \in L(\mathscr{D}(\mathbb{R}))$ is the derivative of Φ at the point $\varphi \in \mathscr{D}(\mathbb{R})$.

If $\Phi \in \mathscr{D}'(\mathbb{R})$, then one recovers the usual definition of Schwartz's distribution theory. If $\Phi \in \mathscr{E}(\mathscr{E}'(\mathbb{R}))$, then φ ranges in $\mathscr{E}'(\mathbb{R})$ and $((d/dx)\Phi) \in \mathscr{E}(\mathscr{E}'(\mathbb{R}))$.

1.4. Proposition. *If $\Phi \in \mathscr{E}(\mathscr{E}'(\mathbb{R}))$, then $\mathscr{A}((d/dx)\Phi) = (d/dx)(\mathscr{A}\Phi)$, that is, the x derivative in $\mathscr{E}(\mathscr{E}'(\mathbb{R}))$ corresponds, via $\mathscr{A}$, to the derivative in $\mathscr{E}(\mathbb{R})$.*

Proof. If $\Phi \in \mathscr{E}(\mathscr{E}'(\mathbb{R}))$,

$$\left(\mathscr{A}\left(\frac{d}{dx}\Phi\right)\right)(x) = \left(\frac{d}{dx}\Phi\right)(\delta_x) = -\Phi'(\delta_x)\left(\frac{d}{dx}\delta_x\right).$$

We have

$$\left(\frac{d}{dx}(\mathscr{A}\Phi)\right)(x) = \lim_{\substack{\zeta\to 0\\ \zeta\in\mathbb{R}-\{0\}}} \frac{1}{\zeta}(\Phi(\delta_{x+\zeta}) - \Phi(\delta_x)).$$

From the mean value theorem applied to $\Phi \in \mathscr{E}(\mathscr{E}'(\mathbb{R}))$,

$$\left(\frac{d}{dx}(\mathscr{A}\Phi)\right)(x) = \lim_{\substack{\zeta\to 0\\ \zeta\in\mathbb{R}-\{0\}}} (\Phi'(\delta_x))\frac{(\delta_{x+\zeta} - \delta_x)}{\zeta},$$

$$\frac{1}{\zeta}(\delta_{x+\zeta} - \delta_x)(\varphi) = \frac{1}{\zeta}(\varphi(x+\zeta) - \varphi(x)) \to \left(\frac{d}{dx}\varphi\right)(x) \qquad \text{if} \quad \zeta \to 0$$

and this convergence is uniform for any fixed x when φ ranges in any bounded subset of $\mathscr{E}(\mathbb{R})$. Therefore

$$\frac{1}{\zeta}(\delta_{x+\zeta} - \delta_x) \to -\frac{d}{dx}\delta_x$$

in $\mathscr{E}'(\mathbb{R})$ if $\zeta \to 0$ and for any fixed x. Therefore

$$\left(\frac{d}{dx}(\mathscr{A}\Phi)\right)(x) = -\Phi'(\delta_x))\left(\frac{d}{dx}\delta_x\right). \quad \blacksquare$$

1.5. *Remark.* From Definition 1.3 one computes immediately the successive derivatives $(d^k/dx^k)\Phi$, $k = 2, 3,\ldots$, and one has the formula for the derivation of a product $(d/dx)(\Phi_1\Phi_2) = ((d/dx)\,\Phi_1)\,\Phi_2 + \Phi_1((d/dx)\,\Phi_2)$, therefore Leibnitz formula for the computation of $(d^k/dx^k)(\Phi_1\Phi_2)$ holds.

2. Construction of the Algebra $\mathscr{G}(\mathbb{R})$ of Generalized Functions on $\mathbb{R}$

2.1. Definition. If $\Phi \in \mathscr{E}(\mathscr{D}(\mathbb{R}))$, we say that Φ is *moderate* if for every compact subset K of $\mathbb{R}$ and every $k = 0, 1, 2,\ldots$, there is an $N \in \mathbb{N}$ such that

$$\forall \varphi \in \mathscr{A}_N \ \exists c > 0 \ \text{ and } \ \eta > 0 \quad \text{such that} \quad \left| \left(\frac{d^k}{dx^k}\Phi\right)(\varphi_{\varepsilon,x}) \right| \leqslant c\left(\frac{1}{\varepsilon}\right)^N$$

if $x \in K$ and $0 < \varepsilon < \eta$.

We denote by $\mathscr{E}_M(\mathscr{D}(\mathbb{R}))$ the subalgebra of the moderate elements of $\mathscr{E}(\mathscr{D}(\mathbb{R}))$. We clearly have $\mathscr{E}(\mathscr{E}'(\mathbb{R})) \subset \mathscr{E}_M(\mathscr{D}(\mathbb{R}))$.

2.2. Proposition. *Every distribution is moderate.*

Proof. Since this is a local property it suffices to check that any derivative (in the distribution sense) of a continuous function g is moderate.

$$\langle g^{(p)}, \varphi_{\varepsilon,x}\rangle = (-1)^p \int g(\lambda)\frac{1}{\varepsilon}\frac{d^p}{d\lambda^p}\left[\varphi\left(\frac{\lambda - x}{\varepsilon}\right)\right] d\lambda,$$

$$|\langle g^{(p)}, \varphi_{\varepsilon,x}\rangle| \leqslant \left(\frac{1}{\varepsilon}\right)^p \int |g(x+\varepsilon\mu)|\,|\varphi^{(p)}(\mu)|\,d\mu \leqslant \frac{c}{(\varepsilon)^p}.$$

2.3. *Remark.* There are elements of $\mathscr{E}(\mathscr{D}(\mathbb{R}))$ that are not moderate (for instance, $\psi \to \exp(\int (\psi(x))^2\,dx)$, $\psi \in \mathscr{D}(\mathbb{R})$) and moderate elements of $\mathscr{E}(\mathscr{D}(\mathbb{R}))$ that are not distributions (for instance, $\psi \to \int (\psi(x))^n\,dx$, $n > 1$).

Now we want to consider an ideal of $\mathscr{E}_M(\mathscr{D}(\mathbb{R}))$ containing $\ker \mathscr{A}$. One might consider the ideal spanned by $\ker \mathscr{A}$, but the following larger ideal is more useful:

2.4. Definition. We set

$$\mathscr{N} = \Big\{ \Phi \in \mathscr{E}_M(\mathscr{D}(\mathbb{R})) \text{ such that for every compact set } K \text{ in } \mathbb{R}$$
$$\text{every } k = 0, 1, 2,\ldots \text{ there is an } N \in \mathbb{N} \text{ such that,}$$
$$\forall \varphi \in \mathscr{A}q \text{ with } q \geqslant N,\ \exists c > 0 \text{ and } \eta > 0 \text{ such that}$$
$$\left| \left(\frac{d^k}{dx^k}\Phi\right)(\varphi_{\varepsilon,x}) \right| \leqslant c(\varepsilon)^{q-N} \text{ if } x \in K \text{ and } 0 < \varepsilon < \eta \Big\}.$$

Clearly, $\mathscr{N}$ is an ideal of $\mathscr{E}_M(\mathscr{D}(\mathbb{R}))$, and from Proposition 1.2. $\ker \mathscr{A} = \mathscr{N} \cap \mathscr{E}(\mathscr{E}'(\mathbb{R}))$. Any x derivative of an element of $\mathscr{N}$ is still in $\mathscr{N}$.

2.5. DEFINITION. We have

$$\mathscr{G}(\mathbb{R}) = (\mathscr{E}_M(\mathscr{D}(\mathbb{R}))/\mathscr{N}).$$

Here $\mathscr{G}(\mathbb{R})$ is an algebra and any x derivative of an element of $\mathscr{G}(\mathbb{R})$ is still in $\mathscr{G}(\mathbb{R})$. Also $\mathscr{E}(\mathbb{R})$ is a subalgebra of $\mathscr{G}(\mathbb{R})$ since $\mathscr{E}(\mathbb{R}) = (\mathscr{E}(\mathscr{E}'(\mathbb{R}))/\ker \mathscr{A})$ and $\ker \mathscr{A} = \mathscr{N} \cap \mathscr{E}(\mathscr{E}'(\mathbb{R}))$.

2.6. PROPOSITION. *The canonical map from $\mathscr{D}'(\mathbb{R})$ into $\mathscr{G}(\mathbb{R})$ is injective; that is, $\mathscr{D}'(\mathbb{R})$ is canonically contained in $\mathscr{G}(\mathbb{R})$.*

Proof. We prove that $\mathscr{D}'(\mathbb{R}) \cap \mathscr{N} = \{0\}$. If $T \in \mathscr{D}'(\mathbb{R}) \cap \mathscr{N}$ and if $\psi \in \mathscr{D}(\mathbb{R})$, we prove that $\langle T, \psi\rangle = 0$. For fixed ε and φ the function $x \to \langle T, \varphi_{\varepsilon,x}\rangle$ is C^∞ on $\mathbb{R}$ and we set

$$I = \int \langle T, \varphi_{\varepsilon,x}\rangle\, \psi(x)\, dx - \langle T, \psi\rangle.$$

Then

$$\begin{aligned} I &= \langle T_\lambda \otimes \psi_x, \varphi_{\varepsilon,x}(\lambda)\rangle - \langle 1_\lambda \otimes T_x, \varphi_{\varepsilon,x}(\lambda)\, \psi(x)\rangle \\ &= \langle T_\lambda \otimes 1_x - 1_\lambda \otimes T_x, \varphi_{\varepsilon,x}(\lambda)\, \psi(x)\rangle. \end{aligned}$$

The change of variable $\lambda \to \mu = (\lambda - x)/\varepsilon$ gives

$$I = \langle 1_\mu \otimes T_x, \varphi(\mu)(\psi(x - \varepsilon\mu) - \psi(x))\rangle.$$

Then $1_\mu \otimes T_x \in \mathscr{D}'(\mathbb{R}^2)$ and the function f_ε defined by

$$f_\varepsilon(\mu, x) = \varphi(\mu)(\psi(x - \varepsilon\mu) - \psi(x))$$

is in $\mathscr{D}(\mathbb{R}^2)$. $f_\varepsilon \to 0$ in $\mathscr{D}(\mathbb{R}^2)$ if $\varepsilon \to 0$, therefore $I \to 0$ if $\varepsilon \to 0$ and

$$\langle T, \psi\rangle = \lim_{\varepsilon \to 0} \int \langle T, \varphi_{\varepsilon,x}\rangle\, \psi(x)\, dx.$$

It suffices to use $T \in \mathscr{N}$, choosing $\varphi \in \mathscr{A}q$ with q large enough. ∎

3. A CONNECTION BETWEEN NEW AND CLASSICAL PRODUCTS

In [13] Schwartz proves that there does not exist an algebra A such that $\mathscr{C}(\mathbb{R})$ is a subalgebra of A, the function 1 is a unit element in A, the elements of A are "C^∞" with respect to a derivation that coincides with the usual one in $C^1(\mathbb{R})$, and such that the usual formula for the derivation of a product holds. As a consequence our multiplication in $\mathscr{G}(\mathbb{R})$ does not coincide with the usual multiplication of continuous functions, although it does for C^∞ functions. We denote by $\odot$ the product in $\mathscr{G}(\mathbb{R})$ when we fear a confusion with classical products. As an exercise we first check (in particular cases) the differences between new and classical products.

3.1. *Remark.*

$$\mathscr{C}(\mathbb{R}) \ni x\,|x| \neq x \odot |x| \in \mathscr{G}(\mathbb{R}),$$

$$\mathscr{D}'(\mathbb{R}) \ni x\delta_0 = 0 \neq x \odot \delta_0 \in \mathscr{G}(\mathbb{R}).$$

Proof.

$$(x\,|x|)(\varphi_{\varepsilon,x}) = \int \lambda\,|\lambda|\,\frac{1}{\varepsilon}\,\varphi\left(\frac{\lambda-x}{\varepsilon}\right) d\lambda = \int (x+\varepsilon\mu)\,|x+\varepsilon\mu|\,\varphi(\mu)\,d\mu$$

$$(x \odot |x|)(\varphi_{\varepsilon,x}) = \int \lambda\,\frac{1}{\varepsilon}\,\varphi\left(\frac{\lambda-x}{\varepsilon}\right) d\lambda \int |\lambda|\,\frac{1}{\varepsilon}\,\varepsilon\left(\frac{\lambda-x}{\varepsilon}\right) d\lambda$$

$$= \int (x+\varepsilon\mu)\,\varphi(\mu)\,d\mu \int |x+\varepsilon\mu|\,\varphi(\mu)\,d\mu.$$

At $x=0$ the difference between the two expressions is

$$d = \varepsilon^2 \left[\int \mu\,|\mu|\,\varphi(\mu)\,d\mu - \int \mu\varphi(\mu)\,d\mu \times \int |\mu|\,\varphi(\mu)\,d\mu\right].$$

If $\varphi \in \mathscr{A}_1$,

$$d = \varepsilon^2 \int \mu\,|\mu|\,\varphi(\mu)\,d\mu.$$

From the proof in Section 1 that $\mathscr{A}q$ is nonvoid there is a $\varphi \in \mathscr{A}q$ with $\int \mu\,|\mu|\,\varphi(\mu)\,d\mu = 1$, and therefore $d = \varepsilon^2$ for this φ. Therefore $x\,|x| - x \odot |x| \notin \mathscr{N}$ (if not, d would decrease like ε^{q-N} if $\varphi \in \mathscr{A}q$, for any φ and any q large enough).

$$(x \odot \delta_0)(\varphi_{\varepsilon,x}) = \int \lambda\varphi_{\varepsilon,x}(\lambda)\,d\lambda\,\varphi_{\varepsilon,x}(0)$$

$$= \int (x+\varepsilon\mu)\,\varphi(\mu)\,d\mu\,\frac{1}{\varepsilon}\,\varphi\left(-\frac{x}{\varepsilon}\right).$$

If $\varphi \in \mathscr{A}_1$,

$$(x \odot \delta_0)(\varphi_{\varepsilon,x}) = x\,\frac{1}{\varepsilon}\,\varphi\left(-\frac{x}{\varepsilon}\right).$$

If $\varphi(-\frac{1}{2}) \neq 0$ (which is possible if $\varphi \in \mathscr{A}q$, see the proof that $\mathscr{A}_q \neq \varnothing$), $\varepsilon_p = (1/p)$ and $x_p = (1/2p)$, we have

$$(x \odot \delta_0)(\varphi_{\varepsilon_p, x_p}) = \tfrac{1}{2}\varphi(-\tfrac{1}{2}),$$

therefore $x \odot \delta_0 \notin \mathscr{N}$. ∎

Now we expose a very important connection between the new and the classical products, that will reconcile them in practice—although they are theoretically different.

3.2. Definition. Let $G \in \mathscr{G}(\mathbb{R})$ and $\Phi \in \mathscr{E}_M(\mathscr{D}(\mathbb{R}))$ be given in the class of G (what follows will be clearly independent on the choice of Φ in the class of G). If for every $\psi \in \mathscr{D}(\mathbb{R})$ the complex number $\int \Phi(\varphi_{\varepsilon,x})\, \psi(x)\, dx$ has a limit when $\varepsilon \to 0$ independent on $\varphi \in \mathscr{A}_q$ for q large enough and if this limit defines a distribution on $\mathbb{R}$ (when ψ ranges in $\mathscr{D}(\mathbb{R})$), we say that the generalized function G admits an associated distribution $\tilde{G}$, defined by

$$\tilde{G}(\psi) = \lim_{\varepsilon \to 0} \int \Phi(\varphi_{\varepsilon,x})\, \psi(x)\, dx.$$

It follows immediately from the proof of Proposition 2.6 that if $T \in \mathscr{D}'(\mathbb{R})$ and if we consider T in $\mathscr{G}(\mathbb{R})$, then T admits an associated distribution which is T itself.

3.3. *Remark.* The generalized function $(\delta_0)^2$ has no associated distribution:

$$\int (\delta_0)^2(\varphi_{\varepsilon,x})\, \psi(x)\, dx = \int \frac{1}{(\varepsilon)^2} \left(\varphi\left(-\frac{x}{\varepsilon}\right)\right)^2 \psi(x)\, dx.$$

If $\psi \equiv 1$ in a 0 neighborhood, and if $\varepsilon > 0$ is small enough this quantity equals

$$\frac{1}{(\varepsilon)^2} \int \left(\varphi\left(-\frac{x}{\varepsilon}\right)\right)^2 dx = \frac{1}{\varepsilon} \int (\varphi(x))^2\, dx$$

which may tend to ∞ when $\varphi \in \mathscr{A}_q$, q arbitrary. ∎

3.4. Theorem. *If f, $g \in \mathscr{C}(\mathbb{R})$, their product $f \odot g$ in $\mathscr{G}(\mathbb{R})$ is a generalized function which admits an associated distribution and this associated distribution is the classical product $fg \in \mathscr{C}(\mathbb{R})$. If $\alpha \in \mathscr{E}(\mathbb{R})$ and $T \in \mathscr{D}'(\mathbb{R})$, their product $\alpha \odot T \in \mathscr{G}(\mathbb{R})$ is a generalized function which admits an associated distribution and this associated distribution is the classical product αT of the distribution theory.*

Proof.

$$\int (f \odot g)(\varphi_{\varepsilon,x})\, \psi(x)\, dx - \int f(x)\, g(x)\, \psi(x)\, dx$$

$$= \int (f(x+\varepsilon\lambda)\, g(x+\varepsilon\mu) - f(x)\, g(x))\, \varphi(\lambda)\, \varphi(\mu)\, \psi(x)\, dx\, d\lambda\, d\mu$$

$$= \int f(x)(g(x-\varepsilon\lambda+\varepsilon\mu)\, \psi(x-\varepsilon\lambda) - g(x)\, \psi(x))\, \varphi(\lambda)\, \varphi(\mu)\, dx\, d\lambda\, d\mu.$$

It suffices to apply the theorem of dominated convergence, and the result holds also, for instance, if $f \in L^{\infty}_{\text{loc}}(\mathbb{R})$ and $g \in \mathscr{C}(\mathbb{R})$.

$$\int (\alpha \odot T)(\varphi_{\varepsilon,x})\, \psi(x)\, dx - \langle T, \alpha\psi \rangle$$

$$= \langle T_x \otimes 1_\lambda \otimes 1_\mu, \varphi(\lambda)\, \varphi(\mu)(\alpha(x-\varepsilon\lambda-\varepsilon\mu)\, \psi(x-\varepsilon\lambda) - \alpha(x)\, \psi(x)) \rangle,$$

hence the result if $\varepsilon \to 0$.

3.5. *Remark.* We are going to see in the next section that this is in fact a particular case of more general products of distributions defined for instance in [9].

3.6. *Comment.* The usual products fg and αT are therfore, so to speak, "projections" on $\mathscr{D}'(\mathbb{R})$ of the generalized functions $f \odot g$ and $\alpha \odot T$, respectively. This reconciles our new product with the classical products, and shows how our new product of distributions generalizes in a reasonable sense the classical products. It follows from this subtlety that our results agree with those of Schwartz [13].

4. A Few Nonclassical Products of Distributions

Various products of distributions which are not considered in Schwartz's distribution theory [12] are used in physics and have been studied by several mathematicians [1, 3–11] by methods of regularizations and passage to the limit. For instance, the following formulas hold (δ denotes here the Dirac measure δ_0 and $\mathscr{P}/x^n$ denotes the principal value of $1/x^n$)

$$\delta \frac{\mathscr{P}}{x} = -\frac{1}{2}\delta' \tag{1}$$

$$\delta^2 - \frac{1}{\pi^2}\left(\frac{\mathscr{P}}{x}\right)^2 = \frac{1}{\pi^2}\left(\frac{\mathscr{P}}{x^2}\right) \tag{2}$$

from which one obtains other formulas involving the Heisemberg functions δ_+ and δ_- (see [10, 11], for instance). Let us remark that formula (1) is of a very common use in [7]. We are going to notice that the way in which these formulas are obtained is immediately interpretable in the same way as in Section 3 for the more classical products of continuous functions or of a C^∞ function and a distribution: the first members of (1) and (2) are elements of $\mathscr{G}(\mathbb{R})$ which admit an associated distribution which is equal to their second members. Notice that both $(\delta)^2$ and $(\mathscr{P}/x)^2$ are elements of $\mathscr{G}(\mathbb{R})$ that have no associated distribution (see Remark 3.3), and so only the block $\delta^2 - (1/\pi^2)(\mathscr{P}/x)^2$ makes sense within distribution theory.

Mikusinski [9] considers a sequence of functions $\rho_n \in \mathscr{D}(\mathbb{R})$ with:

(a) $\operatorname{supp} \rho_n \to \{0\}$ if $n \to +\infty$,

(b) $\int \rho_n(x)\, dx = 1$,

(c) $\forall k = 0, 1, ..., \operatorname{Sup}_{x \in \mathbb{R}, n \in \mathbb{N}} |(x)^{k+1} \rho_n^{(k)}(x)| \leqslant M_k < +\infty$.

Such a sequence (ρ_n) is called a delta sequence and $\rho_n \to \delta_0$ in $\mathscr{D}'(\mathbb{R})$. Let S and T be two given distributions on $\mathbb{R}$. If for every delta sequence the product $(S * \rho_n)(T * \rho_n) \in \mathscr{E}(\mathbb{R})$ admits a limit in $\mathscr{D}'(\mathbb{R})$ as $n \to +\infty$, we define $ST \in \mathscr{D}'(\mathbb{R})$ as this limit. Then formulas (1) and (2) are checked in [9].

4.1. PROPOSITION. *Let $S, T \in \mathscr{D}'(\mathbb{R})$. If the product ST exists, then the product $S \odot T \in \mathscr{G}(\mathbb{R})$ admits an associated distribution which is ST.*

Proof. From Definition 3.2 one has to check if the formula

$$I_\varepsilon = \int \langle S, \varphi_{\varepsilon,x} \rangle \langle T, \varphi_{\varepsilon,x} \rangle\, \psi(x)\, dx$$

defines the distribution ST when $\varepsilon \to 0$. Setting

$$\rho_\varepsilon(\lambda) = (1/\varepsilon)\, \varphi(-(\lambda/\varepsilon))$$

one obtains a delta sequence (ρ_{ε_n}) for any null sequence (ε_n). Then

$$I_\varepsilon = \int \langle S_\mu, \rho_\varepsilon(x-\mu) \rangle \langle T_\nu, \rho_\varepsilon(x-\nu) \rangle\, \psi(x)\, dx$$

$$I_\varepsilon = \langle (S * \rho_\varepsilon)(T * \rho_\varepsilon), \psi \rangle.$$

Therefore it suffices to apply the definition of ST. ∎

In Itano [8] and Fisher [3–5], the delta sequences are slightly different: our sequences (ρ_{ε_n}) have all the requested properties except that $\rho_{\varepsilon_n} \geqslant 0$ is not true: all the properties they obtain without the use of the positivity of the functions in their delta sequences remain true for us, in particular all main formulas.

5. Nonlinear Functions of Elements of $\mathscr{G}(\mathbb{R})$

If $f \in \mathscr{E}(\mathbb{C}^q)$ (i.e., f is C^∞ on $\mathbb{R}^{2q}$) and if $\Phi_1,\ldots, \Phi_q \in \mathscr{E}(\mathscr{D}(\mathbb{R}))$, $f(\Phi_1,\ldots, \Phi_q)$ is the element of $\mathscr{E}(\mathscr{D}(\mathbb{R}))$ defined by

$$\psi \to f(\Phi_1(\psi),\ldots, \Phi_q(\psi)).$$

We use the classical notation $\mathscr{O}_M(\mathbb{R}^{2q})$ for the space of all C^∞ functions on $\mathbb{R}^{2q}$ (which are slowly increasing) as well as all derivatives.

5.1. Proposition. *If $f \in \mathscr{O}_M(\mathbb{R}^{2q})$ and if $\Phi_1,\ldots, \Phi_q \in \mathscr{E}_M(\mathscr{D}(\mathbb{R}))$, then $f(\Phi_1,\ldots, \Phi_q) \in \mathscr{E}_M(\mathscr{D}(\mathbb{R}))$.*

Proof. Let $\lambda = (\lambda_1,\ldots, \lambda_q) \in (\mathbb{R}^2)^q$ and $|\lambda| = |\lambda_1| + \cdots + |\lambda_q|$. Then there are $p \in \mathbb{N}$ and $c > 0$ with

$$|f(\lambda)| \leqslant c(1 + |\lambda|)^p.$$

From Definition 2.1, if K is a compact subset of $\mathbb{R}$, there is an $N \in \mathbb{N}$ such that if $\varphi \in \mathscr{A}_N$, there exist $\eta > 0$ and $c' > 0$ such that if $1 \leqslant i \leqslant q$, $0 < \varepsilon < \eta$ and $x \in K$ we have

$$|\Phi_i(\varphi_{\varepsilon,x})| \leqslant c'(1/\varepsilon)^N.$$

Therefore

$$|f(\Phi_1(\varphi_{\varepsilon,x}),\ldots, \Phi_q(\varphi_{\varepsilon,x})| \leqslant c''(1/\varepsilon)^{Np}.$$

We have

$$\frac{d}{dx} f(\Phi_1(\psi),\ldots, \Phi_q(\psi)) = \sum_{i=1}^{q} (\partial_i f)(\Phi_1(\psi),\ldots, \Phi_q(\psi)) \left(\Phi_i'(\psi)\left(-\frac{d}{dx}\psi\right)\right)$$

and

$$\Phi_i'(\psi)\left(-\frac{d}{dx}\psi\right) = \left(\frac{d}{dx}\Phi_i\right)(\psi).$$

The verification is the same for all successive x derivatives of $f(\Phi_1,\ldots, \Phi_q)$. ∎

5.2. Definition and Theorem. *If* $q = 1, 2, \ldots$, *if* $f \in \mathcal{O}_M(\mathbb{R}^{2q})$ *and* $G_1, \ldots, G_q \in \mathcal{G}(\mathbb{R})$, *then an element* $f(G_1, \ldots, G_q)$ *of* $\mathcal{G}(\mathbb{R})$ *is well defined as the class of the function* $f(\Phi_1, \ldots, \Phi_q)$ *if* $\Phi_i \in G_i$, $1 \leqslant i \leqslant q$.

If $G_i \in \mathscr{E}(\mathbb{R})$, one recovers the classical definition by letting $\psi = \delta_x$, $x \in \mathbb{R}$. Furthermore since, in fact, all computations are done in $\mathscr{E}(\mathscr{D}(\mathbb{R}))$, after any choice of representatives of the involved generalized functions, it is immediate to check that these new objects satisfy the same rules of computation as their classical analogs. For instance if $G \in \mathcal{G}(\mathbb{R})$ and is "real valued" (obvious definition), then

$$\frac{d}{dx}(\sin G) = (\cos G)\left(\frac{d}{dx} G\right).$$

Proof of 5.2. For simplicity in notations we limit ourselves to the case $q = 1$. If $\Phi_1 - \Phi_2 \in \mathcal{N}$, Φ_1, $\Phi_2 \in \mathscr{E}_M(\mathscr{D}(\mathbb{R}))$, we want to prove that $f(\Phi_1) - f(\Phi_2) \in \mathcal{N}$.

$$\begin{aligned}
&|f(\Phi_1(\varphi_{\varepsilon,x})) - f(\Phi_2(\varphi_{\varepsilon,x}))| \\
&\quad \leqslant \operatorname*{Sup}_{0 \leqslant \theta \leqslant 1} |f'(\Phi_1(\varphi_{\varepsilon,x}) + \theta(\Phi_2(\varphi_{\varepsilon,x}) - \Phi_1(\varphi_{\varepsilon,x})))| \\
&\quad \times |\Phi_1(\varphi_{\varepsilon,x}) - \Phi_2(\varphi_{\varepsilon,x})|.
\end{aligned}$$

If $x \in K$ and $\varepsilon > 0$ is small enough, if $\varphi \in \mathscr{A}_q$ for q large enough

$$\begin{aligned}
|f'(x)| &\leqslant c(1 + |x|)^p, \\
|\Phi_i(\varphi_{\varepsilon,x})| &\leqslant c(1/\varepsilon)^{N_1} \qquad \text{if} \quad i = 1, 2, \\
|\Phi_1(\varphi_{\varepsilon,x}) - \Phi_2(\varphi_{\varepsilon,x})| &\leqslant c\varepsilon^{q - N_2}.
\end{aligned}$$

Therefore

$$|f(\Phi_1(\varphi_{\varepsilon,x})) - f(\Phi_2(\varphi_{\varepsilon,x}))| \leqslant c'(1/\varepsilon)^{N_1 p}(\varepsilon)^{q - N_2}$$

and we obtain the desired majorization of Definition 2.4 for $k = 0$. Now we consider $k = 1$,

$$\begin{aligned}
&\frac{d}{dx}[f(\Phi_1) - f(\Phi_2)](\varphi_{\varepsilon,x}) \\
&\quad = f'(\Phi_1(\varphi_{\varepsilon,x}))\left(\left(\frac{d}{dx}\Phi_1\right)(\varphi_{\varepsilon,x})\right) - f'(\Phi_2(\varphi_{\varepsilon,x}))\left(\left(\frac{d}{dx}\Phi_2\right)(\varphi_{\varepsilon,x})\right) \\
&\quad = [f'(\Phi_1(\varphi_{\varepsilon,x})) - f'(\Phi_2(\varphi_{\varepsilon,x}))]\left(\left(\frac{d}{dx}\Phi_1\right)(\varphi_{\varepsilon,x})\right) \\
&\qquad + f'(\Phi_2(\varphi_{\varepsilon,x}))\left(\frac{d}{dx}(\Phi_1 - \Phi_2)(\varphi_{\varepsilon,x})\right).
\end{aligned}$$

Therefore

$$\left|\left(\frac{d}{dx}[f(\Phi_1)-f(\Phi_2)]\right)(\varphi_{\varepsilon,x})\right| \leqslant \operatorname*{Sup}_{0\leqslant\theta\leqslant 1} |f''[\Phi_1(\varphi_{\varepsilon,x})+\theta(\Phi_2-\Phi_1)(\varphi_{\varepsilon,x})]|$$
$$\times |(\Phi_1-\Phi_2)(\varphi_{\varepsilon,x})| \times \left|\left(\frac{d}{dx}\Phi_1\right)(\varphi_{\varepsilon,x})\right|$$
$$+|f'(\Phi_2(\varphi_{\varepsilon,x}))| \times |(d/dx)(\Phi_1-\Phi_2)(\varphi_{\varepsilon,x})|.$$

A majorization exactly similar to the one done for $k=0$ gives the result and the same clearly holds for any k. ▮

5.3. THEOREM. *If $f \in \mathscr{O}_M(\mathbb{R}^{2q})$ and $g_1,\ldots, g_q \in \mathscr{E}(\mathbb{R})$, then $f(g_1,\ldots, g_q)$ defined in 5.2 as an element of $\mathscr{G}(\mathbb{R})$ admits an associated distribution which is the usual continuous function $x \to f(g_1(x),\ldots, g_q(x))$.*

The proof is similar to the proof of Theorem 3.4 for the more particular case of multiplication. Therefore Proposition 5.3 shows that the computations on generalized functions still agree with the usual computations on continuous functions if one considers the associated distribution.

6. SOLUTIONS OF SOME DIFFERENTIAL EQUATIONS

A mathematical "object" of a fundamental importance in quantum field theory is the "scattering operator" which describes the results of the interaction between particles. It may be heuristically "defined" by the differential equation

$$\begin{aligned} S'(t) &= -i\mathrm{H}_{\mathrm{I}}^{0}(t)\, S(t), \\ S(-\infty) &= \mathrm{Id}, \end{aligned} \tag{6.1}$$

where Id denotes the identity operator on the Fock space and, where $\mathrm{H}_{\mathrm{I}}^{0}(t)$ is the "interaction Hamiltonian." Before the "adiabatic limit" is taken $\mathrm{H}_{\mathrm{I}}^{0}(t)$ is a distribution on $\mathbb{R}$ in the variable t with compact support and valued as an operator on the Fock space (after adiabatic limit $\mathrm{H}_{\mathrm{I}}^{0}(t)$ is worse). A considerably simplified model is therefore the scalar equation

$$\begin{aligned} X'(t) &= ia(t)\, X(t), \\ X(-\infty) &= X_0 \in \mathbb{C}, \end{aligned} \tag{6.2}$$

where a is a real distribution with compact support. We denote by A the primitive of a—in the sense of distribution theory—which is null in the past of the support of a (the variable t represents the time). Since the function on $\mathbb{R}^2$: $(x, y) \to e^{ix}$ is in $\mathcal{O}_M(\mathbb{R}^2)$ it follows from 5.2 that the "object" e^{iA} is a generalized function in $\mathcal{G}(\mathbb{R})$. Setting $X = X_0 e^{iA}$, that may be improperly written as $X(t) = X_0 e^{i\int_{-\infty}^{t} a(u)\,du}$, X is solution of (6.2). If t lies in the future of the support of a, $X(t)$ equals the complex constant $X_0 e^{i\langle a, 1\rangle}$. When t varies from $-\infty$ to $+\infty$, $X(t)$ varies from the initial condition X_0 to the final value $X_0 e^{i\langle a, 1\rangle}$ and in between X is a generalized function.

6.3. *Remark.* There exist generalized functions with a null derivative and that are not usual constants: for instance, the class of $\Phi(\psi) = \int (\psi(x))^n\,dx$ if $n > 1$. This means only that, for a Φ as shown, $\Phi(\varphi_{\varepsilon,x})$ does not depend on $x \in \mathbb{R}$. Such objects are implicitly contained in computations of quantum field theory, for instance, the class of

$$\Phi(\psi) = \int (k^2 + m^2)^{1/2}\, e^{ik(\mu - \lambda)} \psi(\lambda)\, \psi(\mu)\, dk\, d\mu\, d\lambda$$

($m > 0$, k, λ, $\mu \in \mathbb{R}$) takes the appearance (by replacing ψ by δ_x) of the "infinite quantity" $\int (k^2 + m^2)^{1/2}\,dk$.

7. Vector Valued Generalized Functions

If E is a nonnecessarily commutative Banach algebra, we define our generalized functions exactly as in the scalar case, but this generalization is much too weak for the applications. A bornological algebra E is a convex bornological vector space ([2, Section 02]) which is an algebra and is such that the product of two bounded sets is still bounded (such algebras are not necessarily bornological inductive limits of Banach algebras). The definitions of the preceding sections are immediately generalized to

7.1. Definition. We say that $\Phi \in \mathcal{E}(\mathcal{D}(\mathbb{R}), E)$ is moderate if for every compact subset K of $\mathbb{R}$ and every $k = 0, 1, 2, \ldots$, there exists an $N \in \mathbb{N}$ such that if $\varphi \in \mathcal{A}_N$, there are a bounded subset B of E and an $\eta > 0$ such that

$$\left(\frac{d^k}{dx^k}\Phi\right)(\varphi_{\varepsilon,x}) \in \left(\frac{1}{\varepsilon}\right)^N B$$

if $x \in K$ and $0 < \varepsilon < \eta$. We denote by $\mathcal{E}_M(\mathcal{D}(\mathbb{R}), E)$ the subalgebra of the moderate elements of $\mathcal{E}(\mathcal{D}(\mathbb{R}), E)$.

7.2. Definition. We set

$$\mathcal{N} = \{\Phi \in \mathcal{E}_M(\mathcal{D}(\mathbb{R}), E)$$

such that for every compact subset K of $\mathbb{R}$ and for every $k = 0, 1, 2, \ldots$, there is an $N \in \mathbb{N}$ such that for every $\varphi \in \mathscr{A}_q$, $q \geqslant N$, there are $\eta > 0$ and a bounded subset B of E such that

$$((d^k/dx^k)\Phi)(\varphi_{\varepsilon,x}) \in (\varepsilon)^{q-N} B$$

if $0 < \varepsilon < \eta$ and $x \in K\}$.

7.3. Definition. We define the E-valued generalized functions by setting

$$\mathscr{G}(\mathbb{R}, E) = \mathscr{E}_M(\mathscr{D}(\mathbb{R}), E)/\mathscr{N}.$$

Then $\mathscr{G}(\mathbb{R}, E)$ is an algebra and $(d^k/dx^k)\,\mathscr{G}(\mathbb{R}, E) \subset \mathscr{G}(\mathbb{R}, E)$ for any $k \in \mathbb{N}$. Assuming E is a complete lcs one extends immediately Proposition 1.2 and the surjectivity of $\mathscr{A}$ (from [2, Theorem 6.2.1], $L(\mathscr{E}'(\mathbb{R}), E) = \mathscr{E}(\mathbb{R})\,\varepsilon E = \mathscr{E}(\mathbb{R}, E)$ and therefore $\mathscr{E}(\mathbb{R}, E)$ is a subalgebra of $\mathscr{G}(\mathbb{R}, E)$.

Important examples are given by quantum field theory, where it is well known that the free fields operators are distributions, not functions. We shall only consider the simplified case of a single scalar Boson field (the following will obviously remain valid for all other Fermian or Boson fields). The Fock space $\mathbb{F}$ is the Hilbertian direct sum

$$\mathbb{F} = \bigoplus_{\substack{n=0 \\ \text{Hilbertian}}}^{+\infty} (L^2(\mathbb{R}^3))^{\hat{\otimes}_\pi n, s},$$

where s means that we consider the symmetrized tensor product. The creation and annihilation operators defined, for instance, in [2, Section 2.8], and denoted, respectively, by $a^+(\psi)$ and $a^-(\psi)$, if $\psi \in \mathscr{S}(\mathbb{R}^3)$, are linear operators on $\mathbb{F}$ and temperate distributions in the variable ψ. Let D_1 be the algebraic direct sum

$$D_1 = \bigoplus_{\substack{n=0 \\ \text{algebraic}}}^{+\infty} (L^2(\mathbb{R}^3))^{\hat{\otimes}_\pi n, s} \subset \mathbb{F}$$

(states with a finite number of particles) equipped with the direct sum bornology [2, 9.1.1]. $L(D_1)$ denotes the algebra of all bounded linear maps form D_1 into D_1, equipped with its natural bornology [2, 0.8.8]. Then if $\psi \in L^2(\mathbb{R}^3)$, it follows obviously from the formulas defining $a^+(\psi)$ and $a^-(\psi)$ that these are in $L(D_1)$ and that, if ψ ranges in a bounded subset of $L^2(\mathbb{R}^3)$, $a^+(\psi)$ and $a^-(\psi)$ range in bounded subsets of $L(D_1)$.

If $\psi \in \mathscr{S}(\mathbb{R}^3)$ and $t \in \mathbb{R}$, the free field operator $A_0(\psi, t)$ is given by the formula

$$2^{1/2}A_0(\psi, t) = a^+(k \to (k^0)^{-1/2}\, e^{ik^0t}(\mathscr{F}\psi)(-k)) \\ + a^-(k \to (k^0)^{-1/2}\, e^{-ik^0t}(\mathscr{F}\psi)(k)), \qquad (7.4)$$

where $k \in \mathbb{R}^3$, $k^0 = ((k_1)^2 + (k_2)^2 + (k_3)^2 + m^2)^{1/2}$ ($m > 0$ is given), and

$$(\mathscr{F}\psi)(k) = (2\pi)^{-3/2} \int_{\mathbb{R}^3} e^{ik\lambda}\psi(\lambda)\, d\lambda.$$

7.5. Proposition. *For each $t \in \mathbb{R}$ and each $n = 0, 1, 2, \ldots$, the function $\psi \to (d^n/dt^n)\, A_0(\psi, t)$ is a moderate element of $L(\mathscr{D}(\mathbb{R}^3), L(D_1)) \subset \mathscr{E}(\mathscr{D}(\mathbb{R}^3), L(D_1))$. Therefore its class defines a generalized function $(d^n/dt^n)\, A_0(\cdot, t) \in \mathscr{G}(\mathbb{R}^3, L(D_1))$.*

Proof. If $\varphi \in \mathscr{A}_1(\mathbb{R}^3)$, k, $x \in \mathbb{R}^3$, we recall that

$$\varphi_{\varepsilon,x}(k) = (1/(\varepsilon)^3)\, \varphi((k - x)/\varepsilon).$$

Therefore one computes easily that

$$2^{1/2}A_0(\varphi_{\varepsilon,x}, t) = a^+(k \to (k^0)^{-1/2}\, e^{ik^0t}e^{-ikx}(\mathscr{F}\varphi)(-\varepsilon k)) \\ + a^-(k \to (k^0)^{-1/2}\, e^{-ik^0t}e^{ikx}(\mathscr{F}\varphi)(\varepsilon k)).$$

One checks immediately that for any fixed $\varphi \in \mathscr{A}_1(\mathbb{R}^3)$, the set of functions $\{\lambda \to (\varepsilon)^{3/2}\, \varphi(\varepsilon\lambda)\}_{0<\varepsilon<1}$ is bounded in $L^2(\mathbb{R}^3)$. Therefore the sets of functions

$$\{k \to (\varepsilon)^{3/2}(k^0)^{-1/2}\, e^{ik^0t}e^{-ikx}(\mathscr{F}\varphi)(-\varepsilon k)\}_{0<\varepsilon<1, x \in \mathbb{R}^3, t \in \mathbb{R}}$$

and

$$\{k \to (\varepsilon)^{3/2}(k^0)^{-1/2}\, e^{-ik^0t}e^{ikx}(\mathscr{F}\varphi)(\varepsilon k)\}_{0<\varepsilon<1, x \in \mathbb{R}^3, t \in \mathbb{R}}$$

are bounded in $L^2(\mathbb{R}^3)$. Therefore there is a bounded subset B of $L(D_1)$ such that

$$A_0(\varphi_{\varepsilon,x}, t) \in (\varepsilon)^{-3/2}\, B$$

if x ranges in $\mathbb{R}^3$ and t in $\mathbb{R}$ (a fortiori for x in a compact subset of $\mathbb{R}^3$ and for fixed t). Now we have to consider the successive x derivatives. This amounts to functions

$$k \to (k_1)^{\alpha_1} \times (k_2)^{\alpha_2} \times (k_3)^{\alpha_3} \times (\mathscr{F}\varphi)(\varepsilon k)$$

for $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{N}$. We have

$$\|k \to (k_1)^{\alpha_1}(k_2)^{\alpha_2}(k_3)^{\alpha_3}(\mathcal{F}\varphi)(\varepsilon k)\|_{L^2(\mathbb{R}^3)}$$
$$= (\varepsilon)^{(-3/2-\alpha_1-\alpha_2-\alpha_3)} \|k \to (k_1)^{\alpha_1}(k_2)^{\alpha_2}(k_3)^{\alpha_3}(\mathcal{F}\varphi)(k)\| L^2(\mathbb{R}^3).$$

Therefore we obtain majorizations analogous to those given and we have the result for $n = 0$. The results for $n = 1, 2, \ldots$, are proved in exactly the same way. ∎

Therefore the multiplication of the free fields operators is now possible, and this result will be applied later to a mathematical explanation of classical heuristic computations of quantum field theory.

Addendum

After he knew of this paper, J. Tysk (from Uppsala, Sweden) proved that other products of distributions defined by Hörmander (Fourier integral operators I, *Acta Math.* **127** (1971), 79–181) and more generally Ambrose (Products of distributions..., *J. Reine Angew. Math.* **315** (1980), 73–91) may be intepreted in the same way as the products considered here in Sections 3 and 4. His proof is very close to the proof of Theorem 3.3.1 in Tysk "On the Multiplication of Distributions," Uppsala Univ., Uppsala, Sweden, 1981.

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References

1. P. Antosik, J. Mikusinski, and R. Sikorski, "Theory of Distributions, the Sequential Approach," Elsevier, Amsterdam/New York, 1973.
2. J. F. Colombeau, "Differential Calculus and Holomorphy: Real and Complex Analysis in Locally Convex Spaces," North-Holland, Amsterdam, 1982.
3. B. Fisher, The product of distributions, *Quart. J. Math. Oxford Ser.* (2) **22** (1971), 291–298.
4. B. Fisher, The product of distributions x^{-r} and $\delta^{(r-1)}(x)$, *Proc. Cambridge Philos. Soc.* **72** (1972), 201–204.
5. B. Fisher, The neutrix distribution product $x_+^{-r}\delta^{(r-1)}x$, *Studia Sci. Math. Hungar.* **9** (1974), 439–441.
6. A. Gonzalez Dominguez and R. Scarfiello, Nota sobre la formula $(v.p.(1/x))\delta = -\frac{1}{2}\delta'$, *Rev. Un. Mat. Argentina* **1** (1956), 53–67.

7. W. HEITLER, "The Quantum Theory of Radiation," Oxford Univ. Press (Clarendon), London/New York (several editions).
8. M. ITANO, Remarks on the multiplicative products of distributions, *Hiroshima Math. J.* **6** (1976), 365–375.
9. J. MIKUSINSKI, On the square of the Dirac delta-distribution, *Bull. Acad. Polon. Sci. Ser. Math. Astronom. Phys.* (14) **9** (1966), 511–513.
10. E. E. ROSINGER, "Distributions and Nonlinear Partial Differential Equations: Weak and Sequential Solutions," North-Holland Math. Studies No. 44, 1980.
12. L. SCHWARTZ, "Théorie des Distributions," Hermann, Paris, 1966.
13. L. SCHWARTZ, Sur l'impossibilité de la multiplication des distributions, *C. R. Acad. Sci. Paris* **239** (1954),. 847–848.